Chapter 2. Comparison

What does it mean to compare?

We compare things every day but do not usually concern ourselves with the meaning of comparison. But when confronting the question one gets confronted with some deep metaphysical ideas. For instance “is 1 and 1 equivalent to 3?” (Fig. 2.1). The answer, as one notices, would depend on the observer. If the observer is making the judgment based upon addition then they are clearly not equivalent. However, they become equivalent if his/her basis is “family of numbers”.

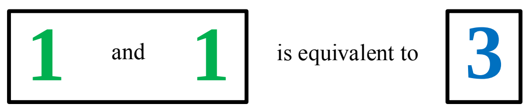


Figure 2.1. The numbers 1 and 1 are equivalent to 3 in the context of family of numbers. However in the context of addition as operation, 1 and 1 is not equivalent to 3 but to 2 (sum).

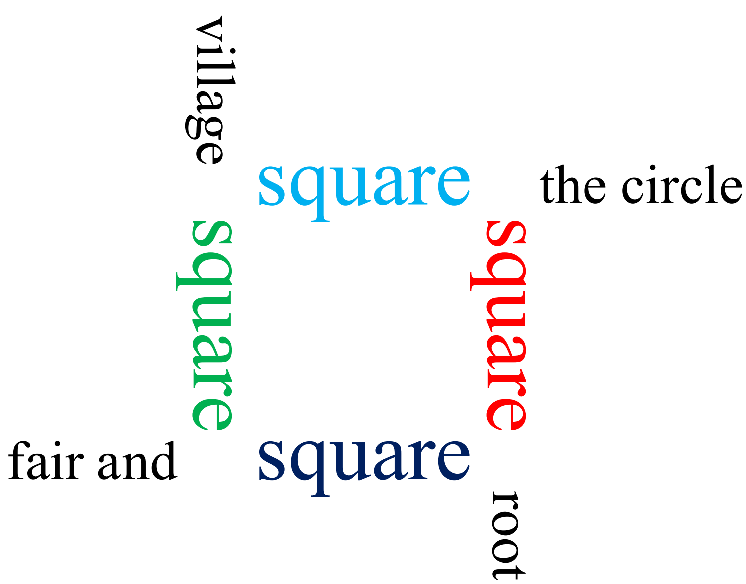


Figure 2.2. The parts of speech of an individual word is determined by the context, i.e., syntax of a given phrase or sentence. In above example (clockwise from top), square is a verb, adjective, adverb and a noun respectively. [Miller, 1991]

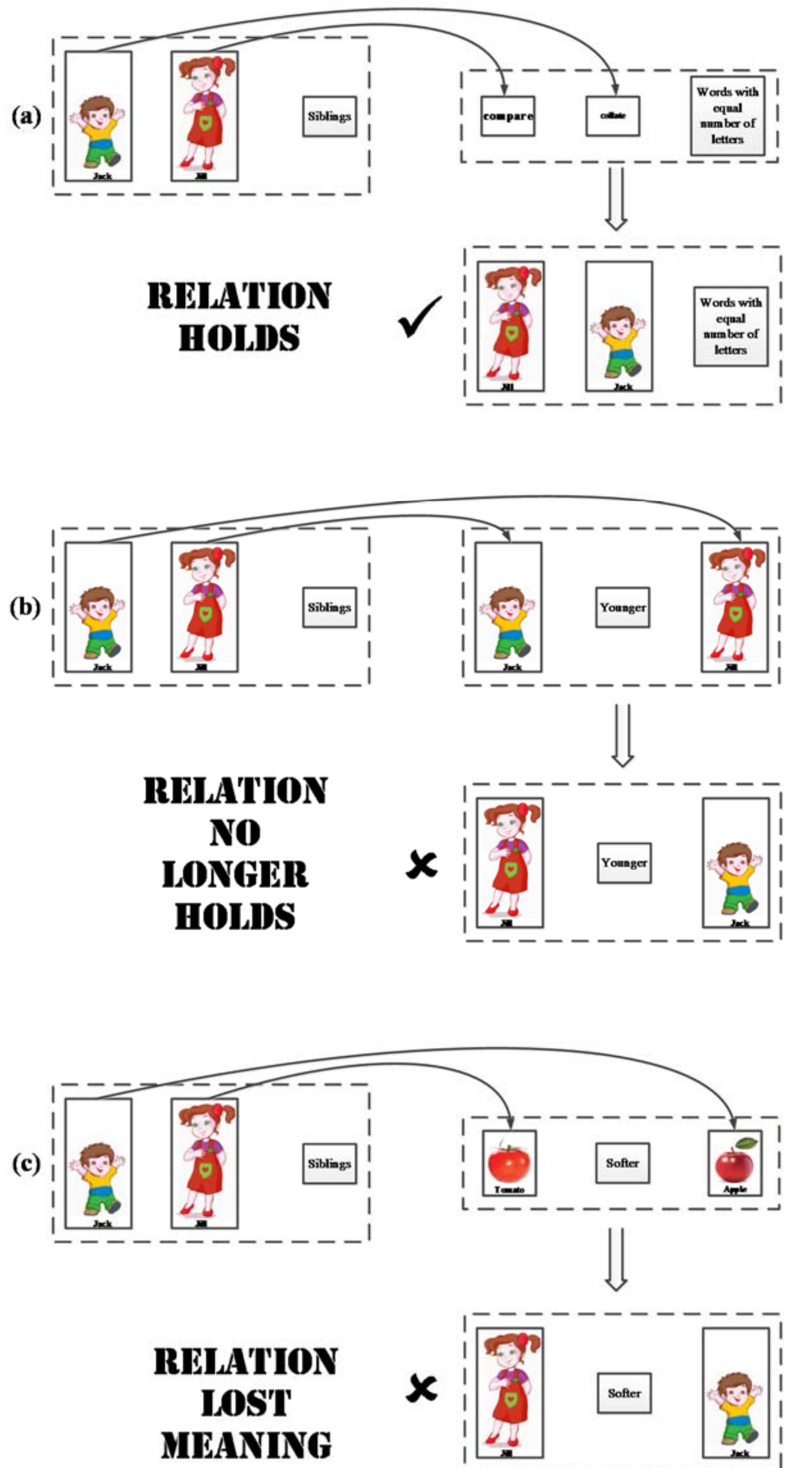


Figure 2.3. Examples demonstrating the three possible outcomes following replacing things from a particular comparison to those in a different comparison. The comparisons or relations between the objects are indicated by respective boxes on the right for some and middle in others. The thing arrows indicate transposition and the large arrows the result after.

Thus comparison of objects depends upon the given situation (context). Figure 2.2 illustrates this where determining the part of speech of the word square depends on the sentence it is associated with.

Observing the above two examples one can then assume that if things in a particular comparison replaces things in a different comparison, then three possible situations are possible [Schreider, 1975]:

- The relation will again hold (Fig. 2.3a).
- The relation will no longer hold (Fig. 2.3b).
- The relation will lose its mean (Fig. 2.3c).

Definition of "compare" and "comparison"

Since subtlety involved in the task of comparison is indicated, before we proceed into a deep discussion let us return to the generic usage of comparison. Figures 2.4, 2.5 and 2.6 shows the various dictionary definitions and usages of the word compare.

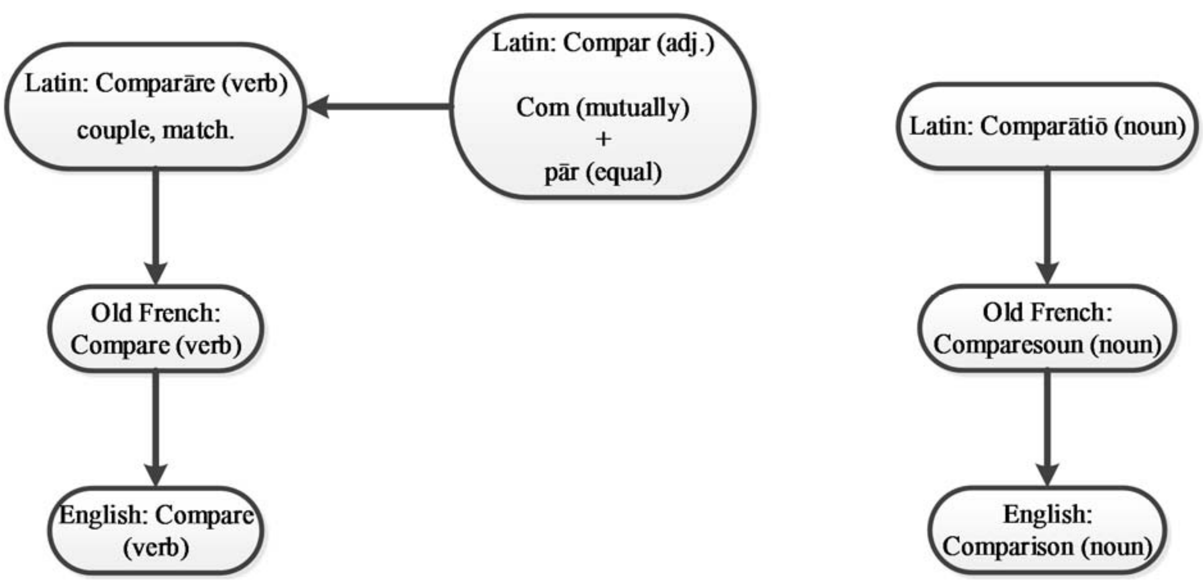


Figure 2.4. Etymology of the words, compare (verb) and comparison (noun) [Ayto, 2005].

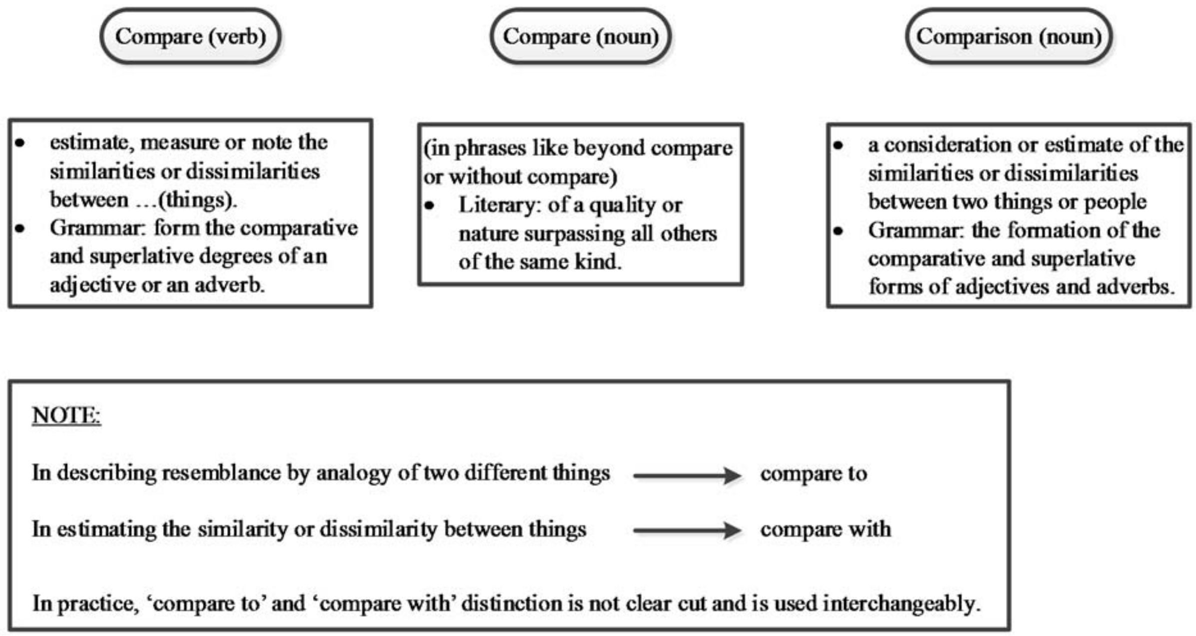


Figure 2.5 Definition and usages of the words compare and comparison [Soanes, 2005].

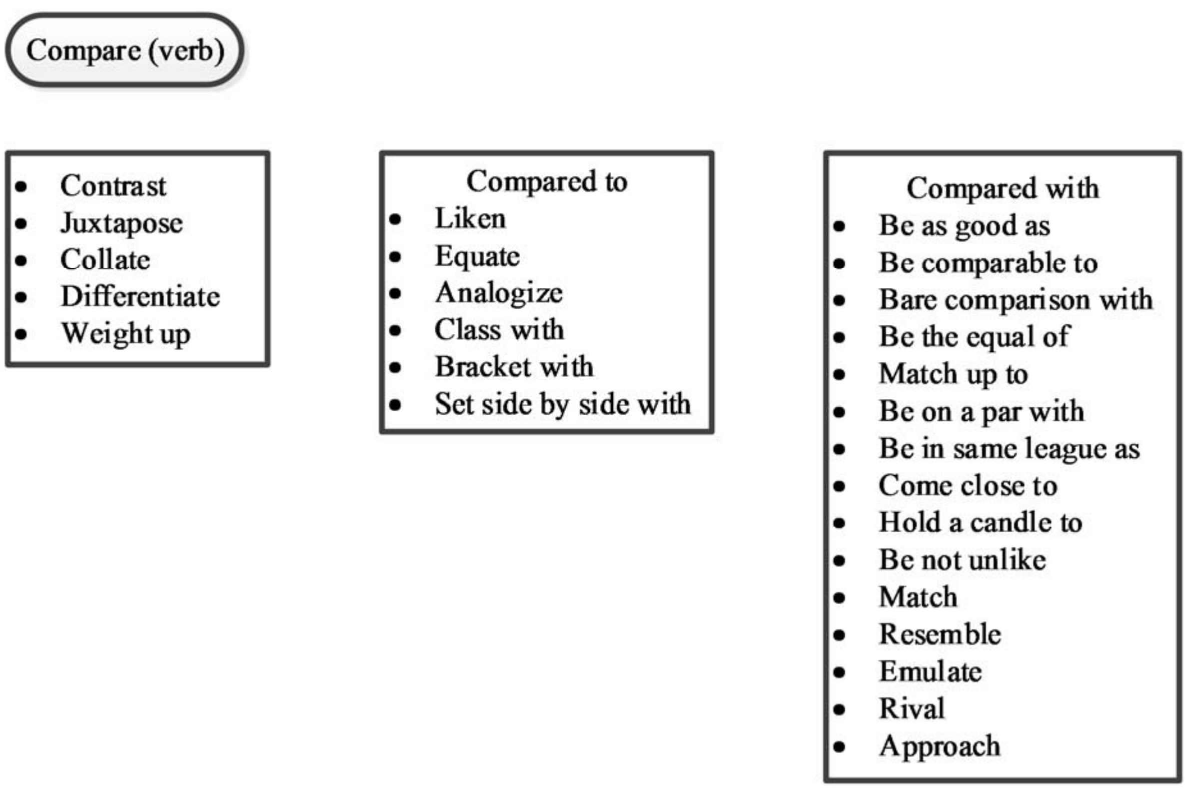


Figure 2.6. Synonyms of the word compare (verb) [Waite et al, 2005].

For our case the word corresponding to ordinary usage is given by the first definitions of the compare (verb) and comparison (noun) in figure 2.5. Synonyms for this defined 'compare' are listed in figure 2.6. Thus, for this usage, compare is to couple, match or estimate the similarities and dissimilarities of things. But the question still remains for the deeper meaning of "what does it mean to compare?" In other words, "how is the comparison between things made?"

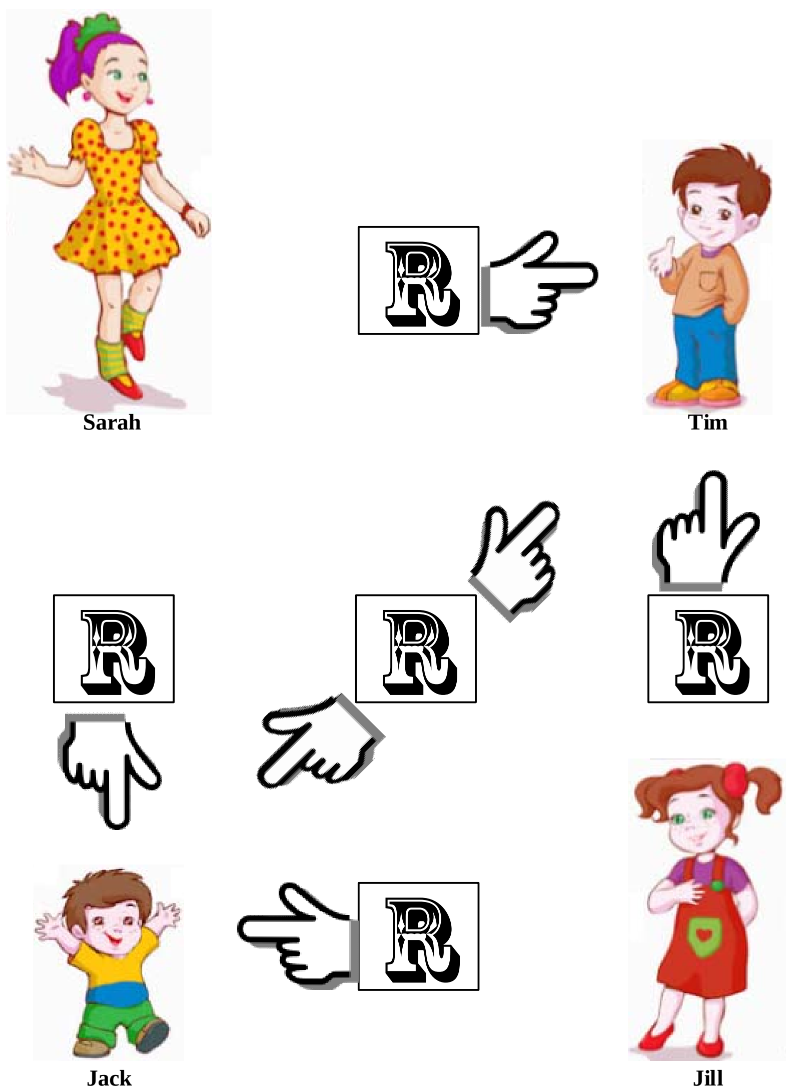


Figure 2.7. An example illustrating the relation (R) "is brother of" indicating only the valid connections for the assumed case that Sarah, Tim, Jill and Jack are children of same parents. Notice that the pointer on either side of R implies that R indicates the objects between which the above relation holds.

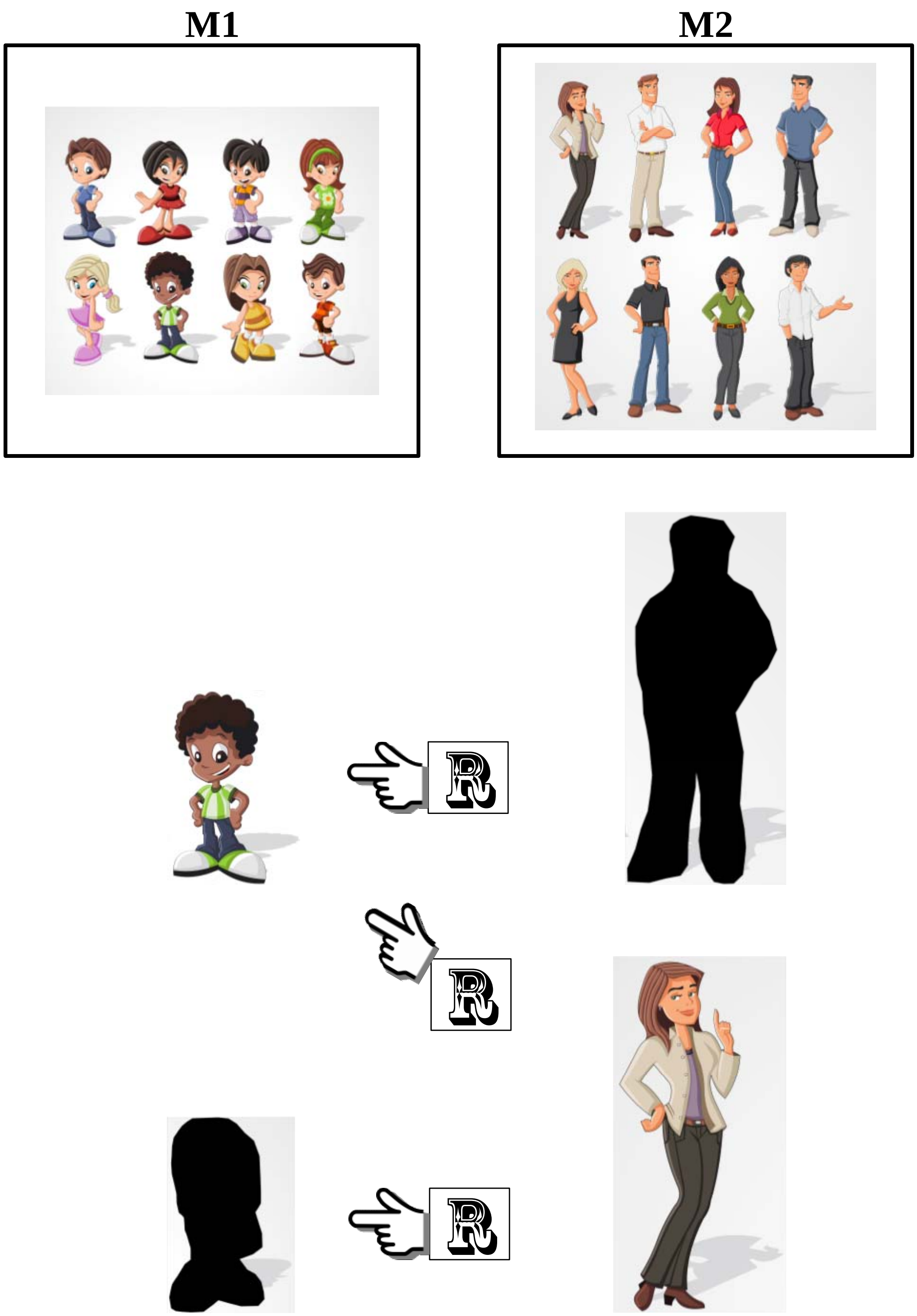


Figure 2.8. An example illustrating the relation (R) “is pupil of” indicating only the valid connections for the assumed case that M1 and M2 are sets of pupils and teachers of the same school. Thus illustrating relations for objects of different sets. The pointer implies that R indicates the objects between which the above relation holds. Silhouettes indicates any.

Giving a "relation"

For a deeper discussion of comparison we will introduce another word here, relation, and consider "how is a relation given?" Understanding the latter will lead us to understanding the former. But before we attempt understanding how a relation is given one's curiosity raises the question of meaning of relation. The verb relate comes from the Latin "relat-" for 'brought back' (from verb referre) and, as the *Oxford English Dictionary* indicates is 'to make or show a connection between' [Soanes, 2005]. Thus the noun relation is 'the way in which two or more people or things are connected' [Soanes, 2005].

Sticking with the everyday meaning of the word relation, giving a relation implies indicating between which objects *the relation* holds. For example let us assume Sarah, Tim, Jill and Jack are children of same parents. Then figure 2.7 shows the valid connections for *the relation* "is brother of". Notice that the children are objects of the same set.

Relations also work for objects of different sets. For example let us assume that a particular school has set of pupils, M1 and set of teachers, M2 (Fig. 2.8). Then, for the relation "is pupil of" a particular pupil is related to a particular teacher or any teacher. Also, that teacher has other pupils. This and the earlier example relates two objects hence binary-relations. Though much of the discussion will be binary-relations one must remember that a relation can have multiple number of objects (Fig. 2.9).

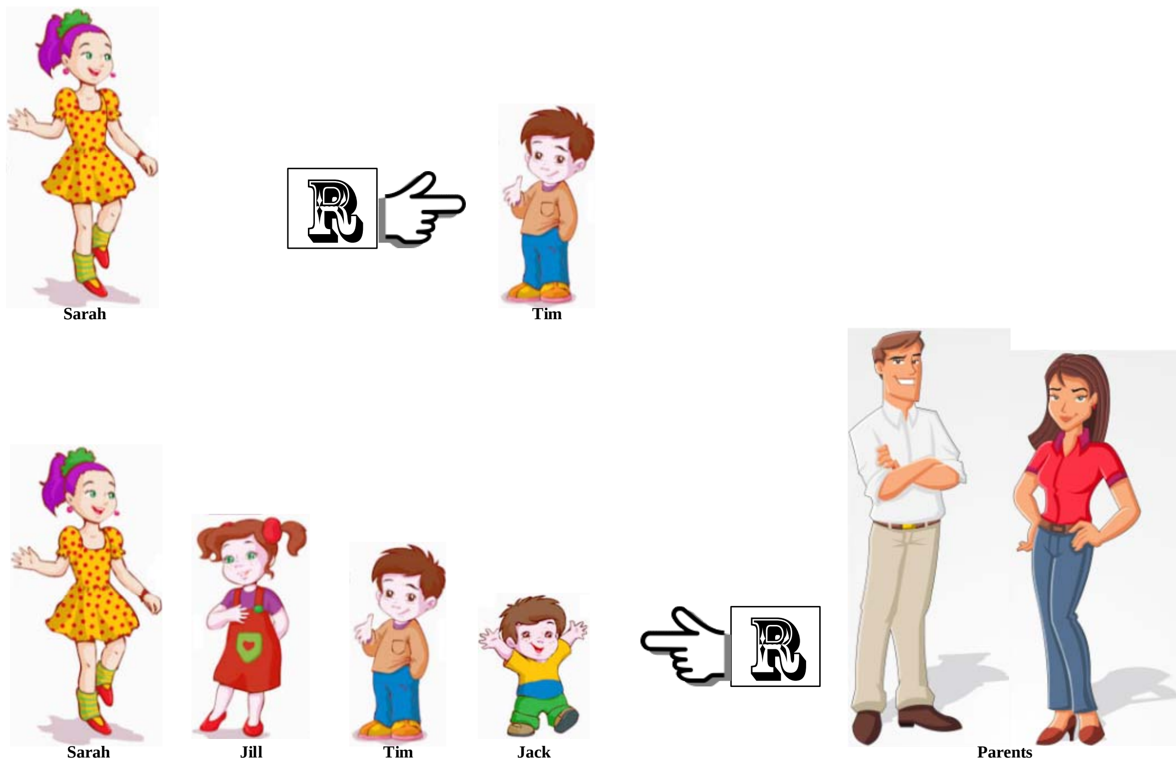


Figure 2.9. Illustration of binary-relation and quaternary-relation. The top example is for relation (R) “is brother of” as seen in Fig. 2.7. The pointer of R indicating that Tim “is brother of” Sarah. Thus relating two object, Tim & Sarah and hence binary-relation.

The bottom example however is for the relation (R) “form children of”. The pointer indicating that Sarah, Jill, Tim and Jack “form children of” same parents, thus relating four objects and hence quaternary-relation. Note that there can be any number of objects for a *given* relation.

Precise definition of “relation”

Following the above qualitative description of what a relation does the next step is to give a formal definition.

Definition 2.1. Given,

M is a set s.t (such that) x, y are any elements belonging to M, $x, y \in$ M;

Ordered pairs $< x, y >$ s.t $< x, y > \neq < y, x >$ unless $x = y$; and

M x M is the set of all ordered pairs $< x, y >$.

Then if a set R is contained in M x M, R $\subset$ M x M, then R is called relation R on the set M.

The informal meaning of above definition is as follows. Let us assume a set of objects (M) and hence set of ordered pairs of the objects (M x M). Then, determined by which pair are connected for a *given relation*, we can choose the subset of M x M, R. In other words if $< x, y > \in$ R then *x* is related by R to *y*, i.e., *x* R *y*. Figure 2.10 illustrates this.

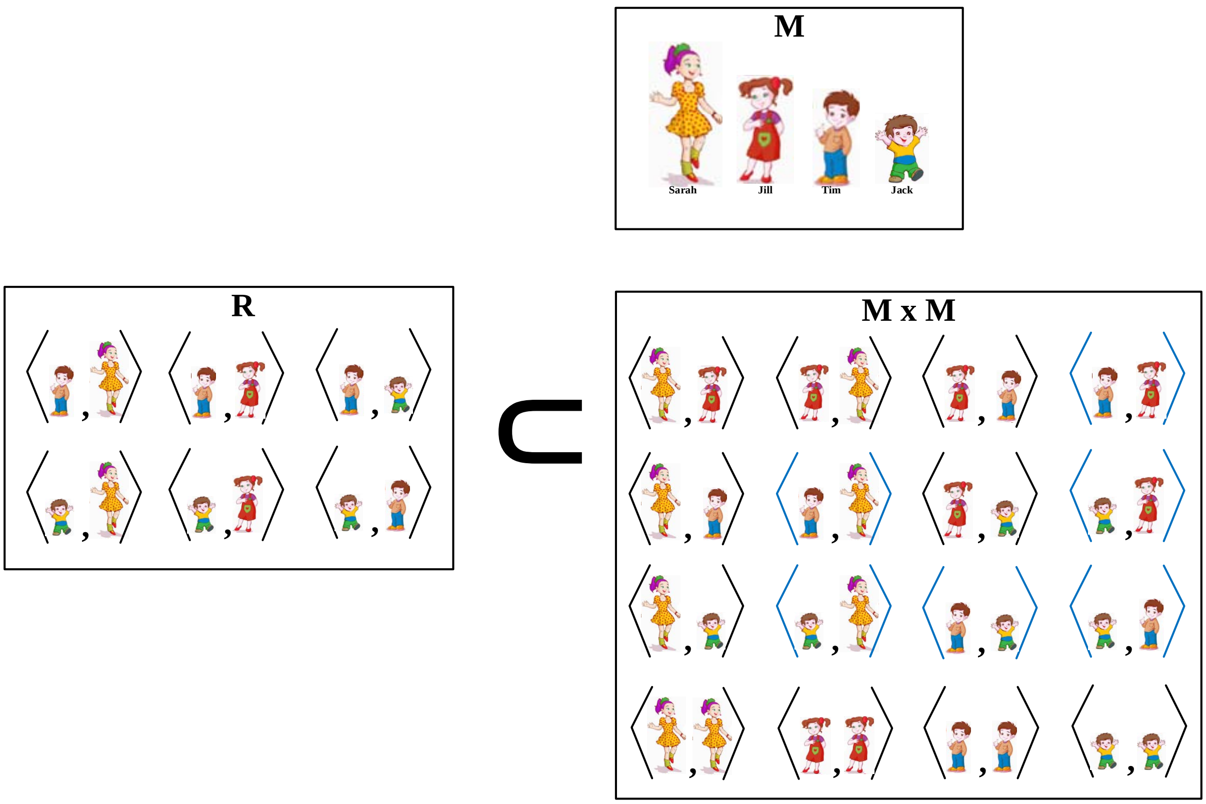


Figure 2.10. An illustrated example of formal definition of relation (Definition 2.1). Let us assume set M to contain the siblings of the same parents and M x M a set of ordered pair of objects. Then for the relation "is brother of" (Fig.2.9) we can construct a set R whose elements are determined from the set M x M (blue brackets) and hence R ⊂ M x M. Note that by definition and for the considered relation $< Tim, Sarah >$ implies Tim "is brother of" Sarah or Tim R Sarah. But $< Tim, Sarah > \neq < Sarah, Tim >$ since the latter does not belong to R ($< Sarah, Tim > \notin$ R), i.e., Sarah "is brother of" Tim does not hold.

The above formal definition of relation is one just form. There are other forms of formally defining representation which all precisely define relation from different perspectives. The curious reader is referred to Schreider's text [Schreider, 1975].

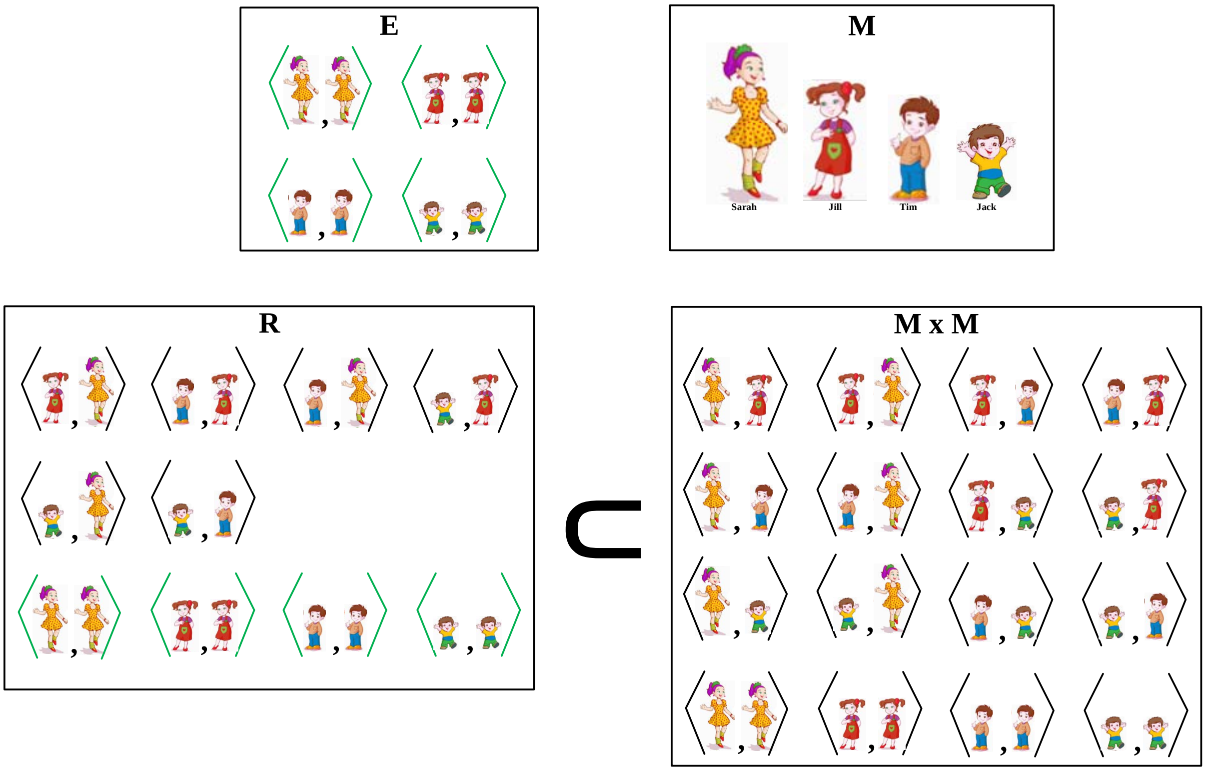


Figure 2.11. An illustrated example of properties of relation (Definition 2.2 & 2.3). Let us consider the same assumptions as Fig. 2.10. For the relation "is not older than" we can determine that Jill "is not older than" Sarah ($< Jill, Sarah >$), Tim "is not older than" Jill ($< Tim, Jill >$) and so on. Hence we can construct the proper subset of M x M, R ⊂ M x M.

Since everybody "is not older than" themselves, all the elements (green bracket) of equality relation (E) is inclusive in the set R, E ⊆ R. Thus, R has the reflexive property.

Some Properties of Relation

This section shall discuss only the properties that would be important for the discussions that would follow. Thus proofs will not be provided here.

Definition 2.2. If x E y is such that x and y are the same element of the set M, then the relation E is called equality relation.

Definition 2.3. If the equality relation E is inclusive in the set R (E ⊆ R), then R is said to have the reflexive property.

Alternatively, a relation R is said to possess reflexivity if it contains ordered pairs $< x, x >$ for every $x \in$ M.

The relation "has the same birthday as" is an example of equality relation. The equality relation E in matrix form will be such that its principal diagonal entries equal one with zero for other entries. Thus E is also called diagonal relation. The relation "is not older than" has the reflexive property because *x* "is not older than" *x* (*x* R *x*) and hence E is included in the relation (Fig.2.11). However for the relation "is brother of" (Fig.2.10), since nobody "is brother of" themselves E ⊈ R and hence is not reflexive. Thus, a relation might not possess reflexivity (Fig.2.12).

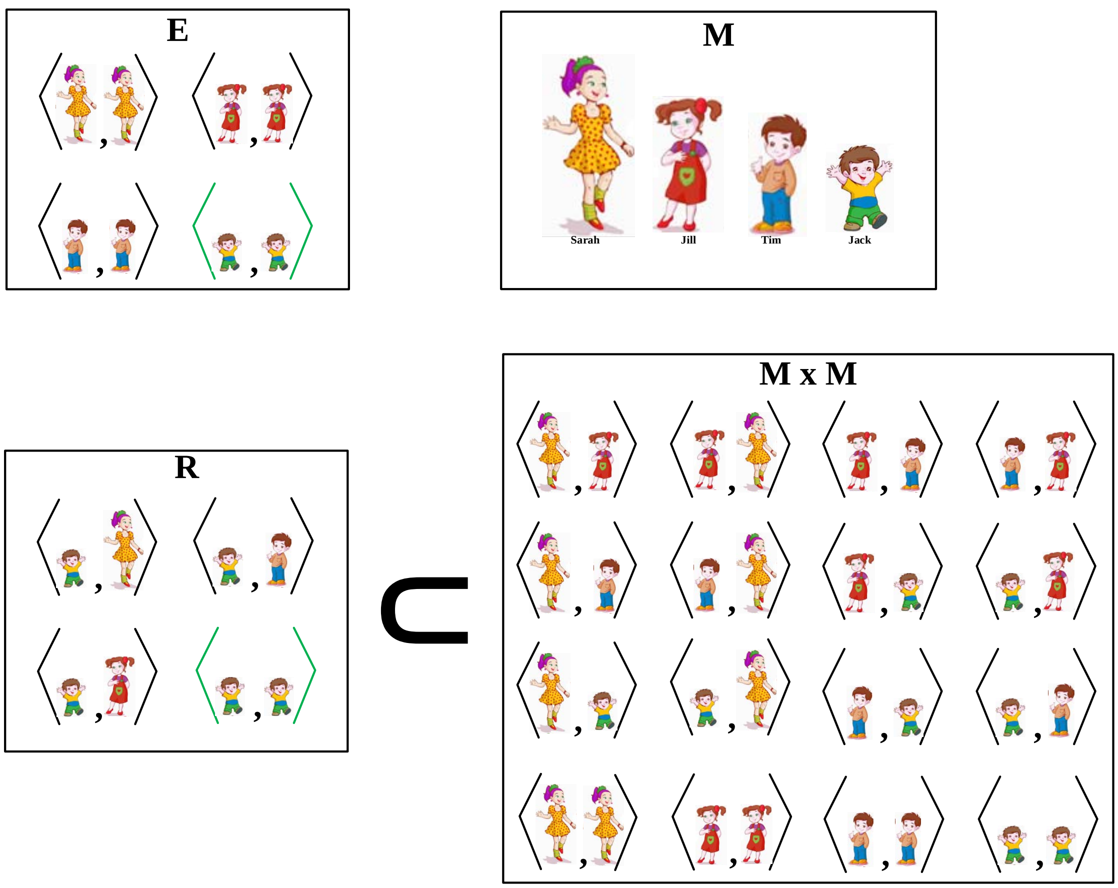
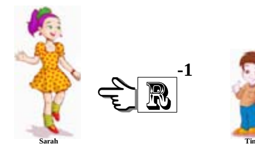


Figure 2.12. An illustrated example of relation not possessing reflexivity. In addition to the earlier assumptions let us also assume that the youngest, Jack is the most popular sibling so much so that others (Sarah, Jill and Tim) are recognized as siblings of the same parent once you identify Jack.

Thus for the relation "is standard for" (siblings of same parent) we can determine that Jack "is the standard for" Sarah, Tim, Jill and himself, hence constructing the proper set, R ⊂ M x M. This relation does not possess reflexivity.

(a)

(b)



(c)

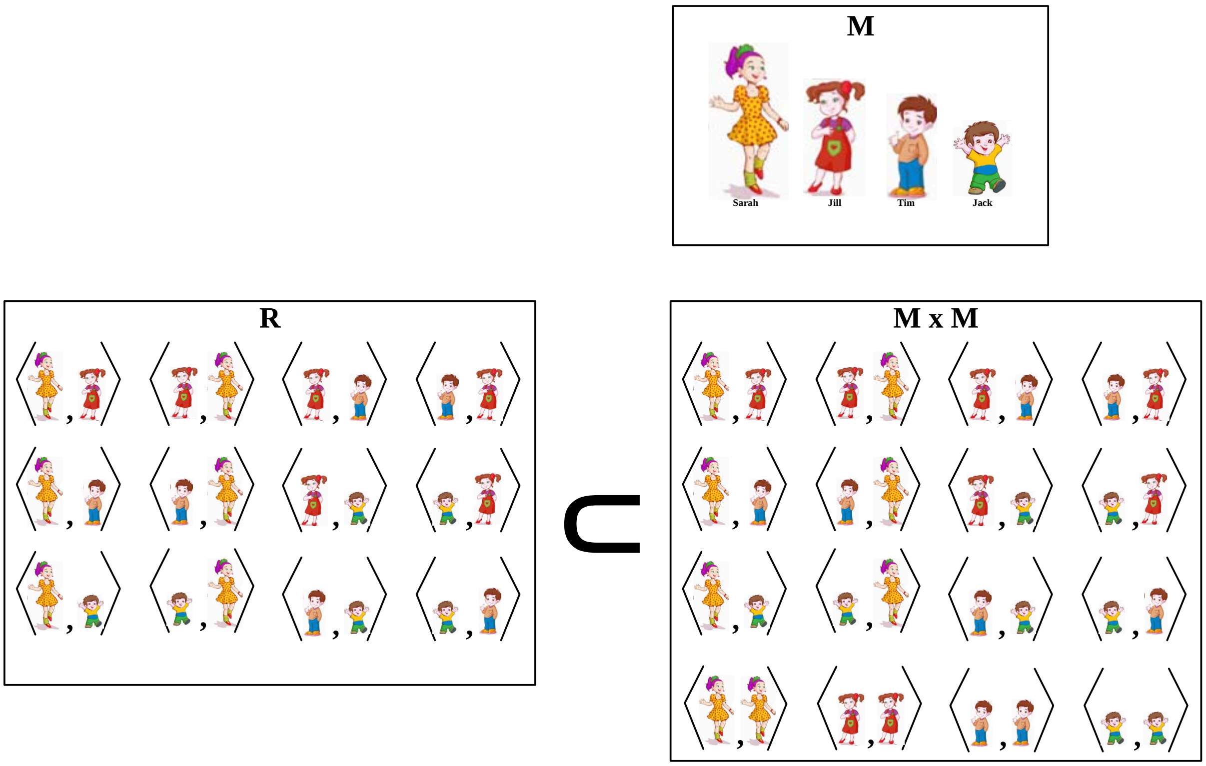


Figure 2.13. Illustrated example for symmetric relation. Let us consider the same assumptions, set of all siblings, M.

(a) Shows the relation “is brother of”. Hence, Tim “is brother of” Sarah, $< Tim, Sarah >$.

(b) The relation “is sister of” is the inverse relation R^{-1} of “is brother of”. Thus, Sarah “is sister of” Tim, $< Sarah, Tim >$. Pointer indicates the first object of respective ordered pair.

(c) For the relation “is sibling of”, we can determine that Sarah “is sibling of” Tim and also Tim “is sibling of” Sarah. Thus, $R = R^{-1}$ and we can construct the proper set, $R \subset M \times M$. This relation will have the symmetric property.

Definition 2.4. Let us assume R is a relation in a set M, x R y. If a relation R^{-1} is defined such that y R^{-1} x, then R^{-1} is called inverse relation.

Definition 2.5. If R ⊆ R^{-1}, then R is called symmetric relation.

Theorem 2.1. A relation R is symmetric if and only if R = R^{-1}.

Alternatively, a relation R is said to possess symmetric property if it contains ordered pairs $< x, y >$ and $< y, x >$.

An example for inverse relation is as follows. If R is the relation "is brother of", i.e., Tim "is brother of" Sarah, Tim R Sarah (Fig.2.13a). Then R^{-1} will be the relation "is sister of", Sarah R^{-1} Tim (Fig.2.13b). However, for the relation "is sibling of", the relations Tim R Sarah and Sarah R Tim holds. Therefore, "is sibling of" has the symmetric property (Fig.2.13c). Notice that if a relation R is symmetric, every element of R is in its inverse.

Definition 2.6. If R ∩ R^{-1} = ∅, then R is called asymmetric relation.

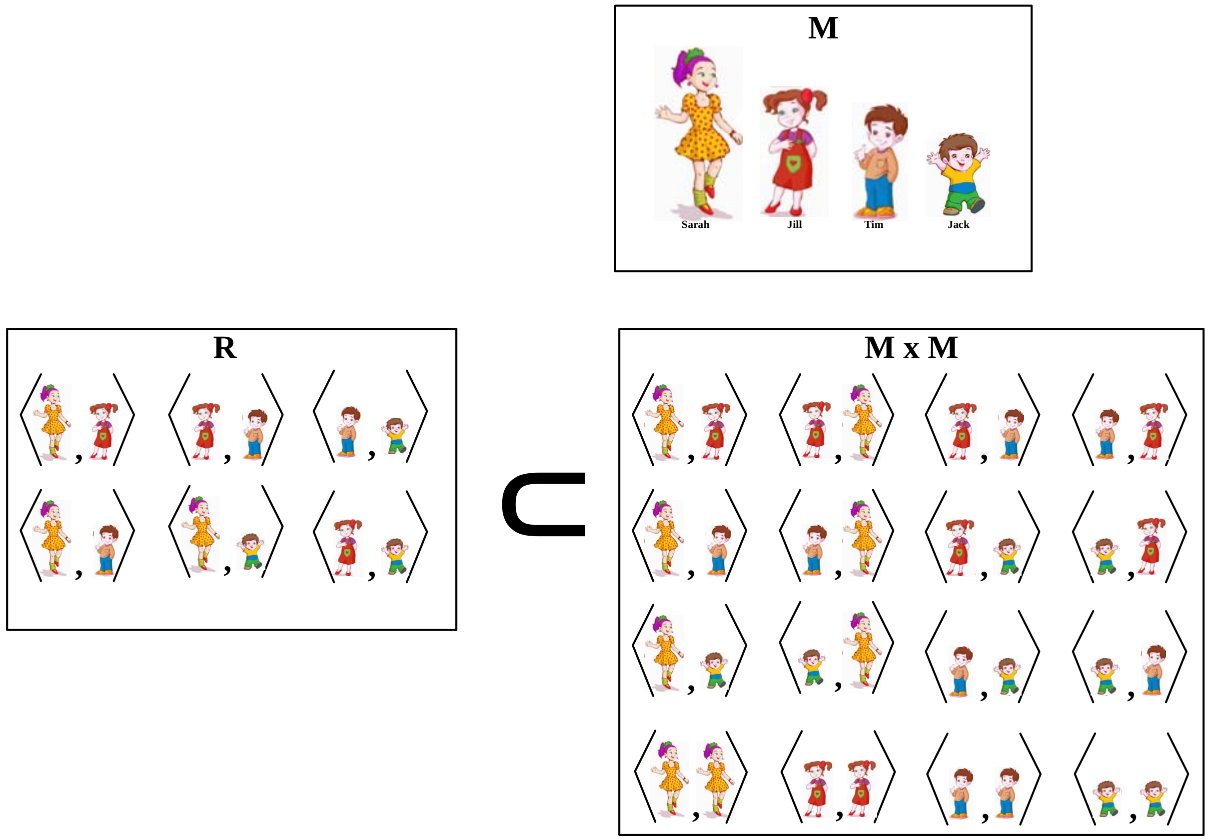
Therefore, the relation "is shorter than" is asymmetric.

Finally, we come to the transitive relation.

Definition 2.7. Given a relation R on a set M, if, for any x, y, z, in M, x R y, y R z implies x R z. Then R is said to be a transitive relation.

Thus for the example "is taller than" (Fig.2.14), Sarah "is taller than" Jill (Sarah R Jill), Jill "is taller than" Tim (Jill R Tim) and Sarah "is taller than" Tim (Sarah R Tim).

(a)



(b)

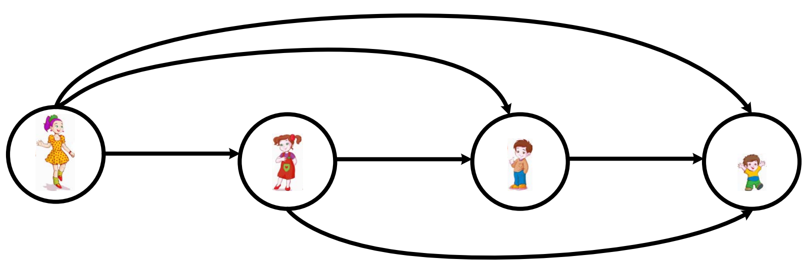


Figure 2.14. Illustration example for transitive property. Let us consider the same assumptions, set of all siblings, M.

(a) For the relation "is taller than", the proper set R is constructed R ⊂ M x M.

(b) The relation set R (or matrix form) may be represented as graphs. The arrows denoting the ordered pairs $< x, y > \in$ R.

For further discussions on comparison the three important properties are the reflexive, symmetric and transitive properties. Illustrating binary relations as network graphs will be helpful in correlating their representations with neural networks. Therefore, the above three properties of relations are summarized using graphs (Fig.2.15).

The illustrated examples shows that comparisons and hence relation between things were possible only after we could single out ordered pairs of things. This identification is necessary for formalizing an understanding of comparison.

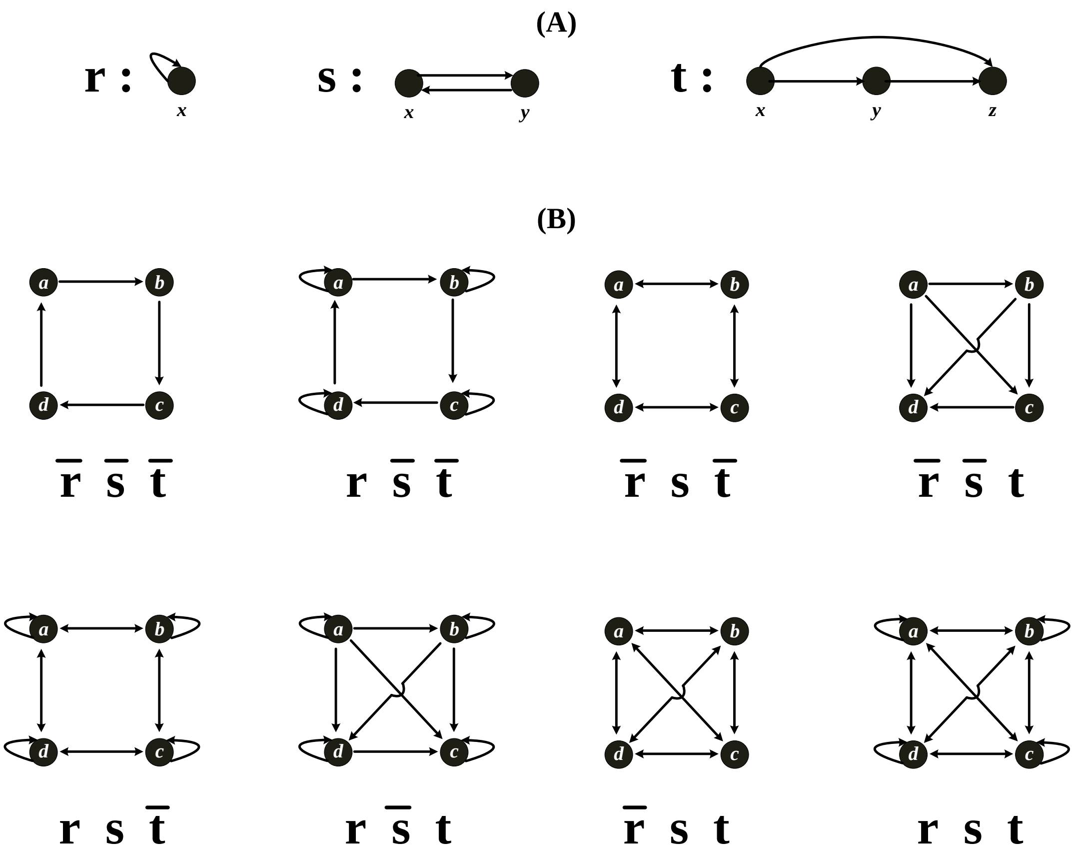


Figure 2.15. Graph illustration of the three relation properties: reflexive (r), symmetric (s) and transitive (t).

(A) Shows the properties individually. Where, **r:** *x* R *x*; **s:** *x* R *y* & *y* R *x*; and **t:** *x* R *y*, *y* R *z* and *x* R *z*.

(B) Shows various combinations of the three properties, from far top left which does not possess any of the three properties ($\bar{r}\ \bar{s}\ \bar{t}$) to bottom right corner possessing all three (***r s t***).

The notion of "identical" (identity)

Just as we considered the ordinary usage of relation let us do the same for the word identical. In everyday usage we say objects are identical without concerning ourselves with the exact meaning of the word. Hence (Fig.2.9), from the point of view of a stranger evaluating in term of physical development and seeing Sarah, Jill, Tim and Jack, they are "all children" and hence identical. However from the point of view of their parents they acquire individuality but again becomes identical when evaluated in terms of development.

We can make three important observations from the above example. They are

1. For a certain set of objects the word "identical" is always understood as a binary-relation. For instance from the point of view of the stranger Sarah "is identical to" Jack;
2. The content of this relation depends on the considered situation or observer passing the judgment with his/her point of view. Thus from the point of view of their parents calling out their names, Sarah "is identical to" Jack no longer holds;
3. The word "identical" is related to the notion of with "interchangeability". From the point of view of the stranger, Jack "is identical to" Sarah also holds true and to him/her Jack and Sarah are interchangeable as "children".

Therefore, it is fair to assume that in a given situation if objects possess same collection of formal features in a particular context then the objects are interchangeable in that context.

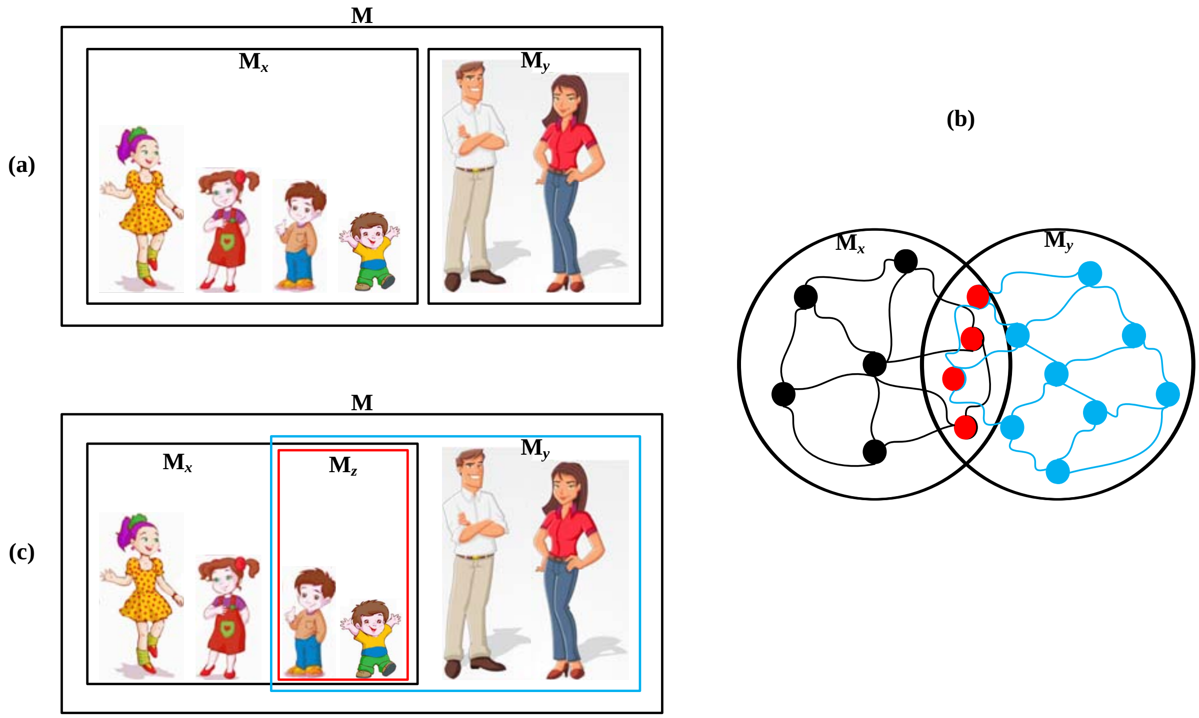


Figure 2.16. Illustration for example set of interchangeability. (a) Considering M as set of family members, if M_x is subset of children and M_y the subset of parents then union of all the subsets is M.

(b) If $M_x \cap M_y \neq \emptyset$ for some M_x, $M_y \subset M$, then there are objects (red) that are common to both M_x & M_y. Since M_x is a particular subset of interchangeable (black lines) objects belonging to M_x, black objects in M_x are interchangeable with red objects. Similarly blue objects in M_y are interchangeable (blue lines) with red objects.

(c) Let us assume M_x is the subset of children, M_y is the subset of parents & sons and M_z, the subset of boys. Then, for the relation "are family member", the boys are interchangeable in M_x ($M_z \subseteq M_x$) and also in M_y ($M_z \subseteq M_y$).

Interchangeability of elements in $M_z(= M_x \cap M_y)$ is conditioned by the context. Thus for this case the family of all the proposition for the interchangeability or "property set" is, {"there exist a set of siblings of same parents", "there exist a set of parents & sons", "set of boys"}.

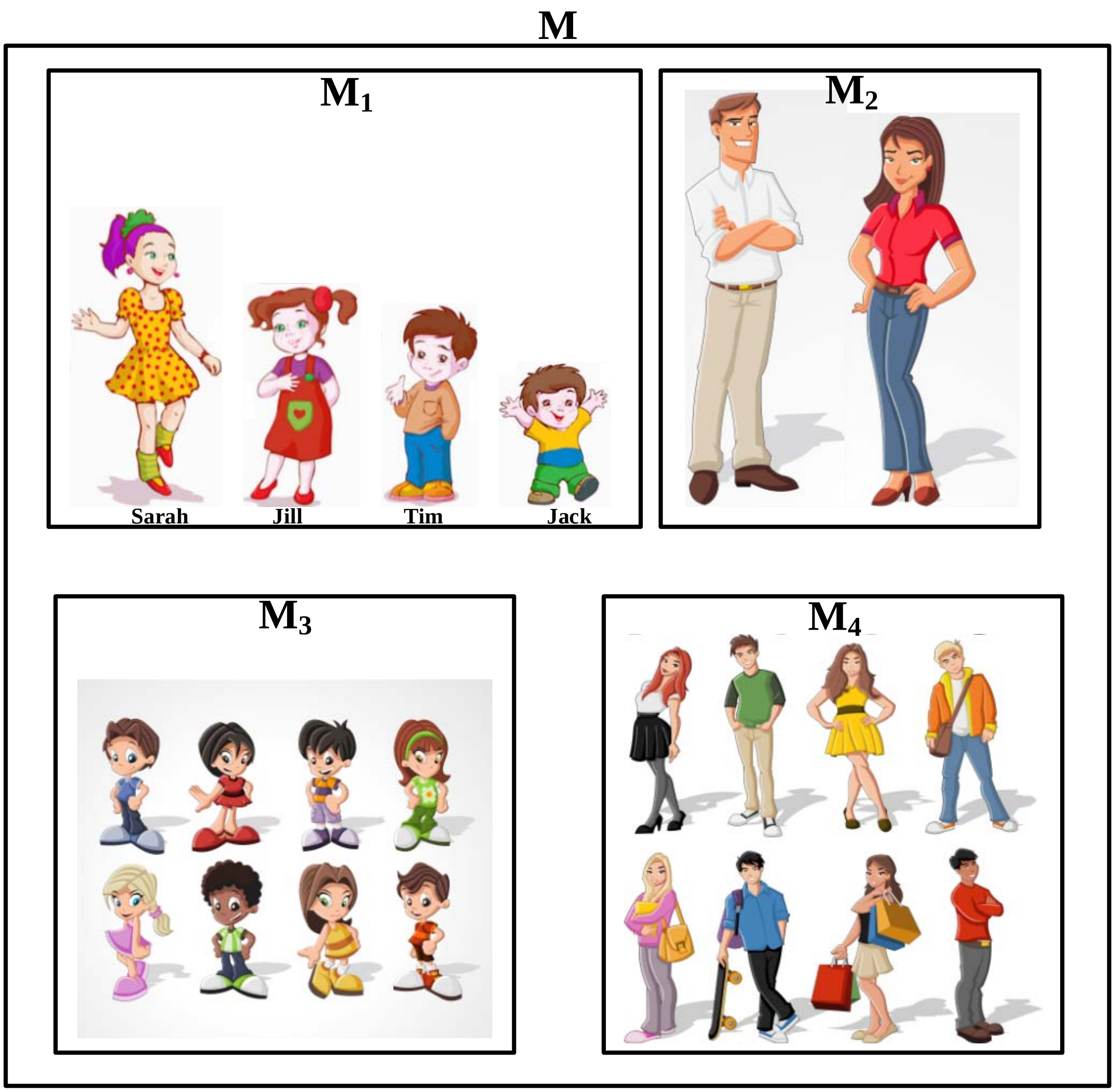


Figure 2.17. Illustration for definitions of equivalence. For the set M of objects (people) let us assume M_1 to be a subset of siblings, M_2 a subset of parents, M_3 a subset of pupils and M_4 a subset of university students. Thus the system of non-empty subsets $\{M_1, M_2, M_3, M_4\}$ of M satisfies the two condition: $M = M_1 \cup M_2 \cup M_3 \cup M_4$ and $M_i \cap M_j = \emptyset$ for $i \neq j$, for these four defined property set specifications. Thus, the system is the partition of M and M_i's are its classes.

For the class M_1 let us consider the relation R, "children with same parents". Then x R y only hold for all people in M_1. This is an equivalence relation. Note that that the relation is reflexive, $< Sarah, Sarah >$, symmetric; $< Sarah, Jill >$ and $< Jill, Sarah >$ and finally transitive, $< Sarah, Jill >$, $< Jill, Tim >$ and $< Sarah, Tim >$.

On the other hand for the relation "are two boys", the relation holds only for some people in M_1 and M_2 and hence no longer specifies an equivalence relation in M_1. It can, however, hold for some subset $M_5 \subset (M_1 \cup M_3)$. If the relation is "are two males" it holds for a subset $M_6 \subset M$ and M_6 is a partitioning of M in which the relation is an equivalence relation.

The notion of "interchangeability"

Let us consider M to be a set of objects such that some of them are interchangeable. Now consider M_x to be a set of all objects x in M that are interchangeable, $x \in M_x$. Thus for all such subsets of M, the union of all M_x is

$M = \bigcup M_x$ (Fig. 2.16a).

Let us now assume $M_x \cap M_y \neq \emptyset$ for some M_x, $M_y \subset M$. Then there must exist some element z (red, Fig.2.16b) belonging simultaneously to both M_x and M_y. Thus, x is interchangeable with z and z is interchangeable with y. This implies set M_z possess some relation-defining property $P \Longrightarrow R'$ such that under R' any $x \in M_x$ is interchangeable with any $x \in M_y$.

Thus, the union $M_x \cup M_y$ comprises a superset M of interchangeable elements within context(s) by which $M_z = M_x \cap M_y$ is interchangeable in both M_x and M_y. The context of the interchangeable property of M is conditioned by the interchangeability context of $M_x \cap M_y$ (Fig.2.16c). This intersection, $M_x \cap M_y$, is said to define a "property set".

The notion of "equivalence"

Now, clearly a singleton object x is identical to itself. Were that not so the notion of an "object" would have no comprehensible meaning. Consider, therefore, two interchangeable objects x and y. If there is some relation R under which x and y can also be said to be identical then we say x and y are **equivalent** and R is called as equivalence relation. The question then become, "what is required for a relation of equivalence to exist?"

We can conclude that the concept of "equivalence" is conditioned by the existence of some property set. Contextual equivalence therefore provides a foundation for higher abstract levels of set inclusions and formal equivalences, and thereby provides a mathematical principle for constructing the concept of **Nature** as a unity of objects in one **Nature**. This, by the way is the principal function of **teleological reflective judgment** in mental physics.

Building upon the preceding discussion of sets and subsets containing interchangeable objects of M, we can make the following observations. If x and y are objects then clearly x R x and y R y are required for the relation. If x and y are interchangeable then it must also hold true that x R y implies y R x because otherwise saying x and y are interchangeable would have no comprehensible meaning. Finally, if x, y, and z are interchangeable objects such that x R y and y R z, then it must likewise be true that x R z. But these are merely the reflexive, symmetric and transitive properties of relation. In order to say x, y, and z are appearances of the same object O it is sufficient to say

> Definition 2.8. An equivalence relation R is a relation that has the reflexive, symmetric, and transitive properties.

Figure 2.17 illustrates this definition.

The possibility of logical partitions is based on equivalence.

> Theorem 2.2. If relation R on set M is reflexive, symmetric and transitive, then there exist a partition $\{M_1, M_2, \ldots\}$ of M such that x R y hold if and only if x and y belong to a common class M_i (in the partition).

The above is in fact a theorem in the mathematics of discrete structures. Again refer figure 2.17 for an illustrated example. Also since an equivalence relation possesses the reflexive, symmetric and transitive property, the bottom right corner of figure 2.15b is a graph of equivalence relation.

It was assumed above that in a given situation interchangeable objects possess one and same collection of formal features. This implies that an object in a particular class contains information about other objects within this class. This idea of information is further developed and explicated using “resemblance” and Christopher Zeeman’s “tolerance” by Schreider [Schreider, 1975].

The reader must have noticed that though “equivalence” has been explicated the question of “how do we get ‘a given situation’?” or context is yet to be answered. To understand this will require a philosophical perspective and hence must consider the metaphysics behind “context”. This is discussed in the next chapter.